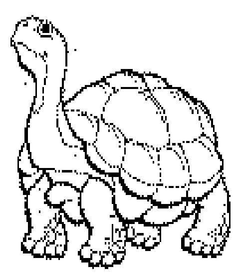
Zeno’s Paradox of the Tortoise and Achilles

Zeno of Elea (*circa* 450 b.c.) is credited with creating several famous paradoxes, but by far the best known is the paradox of the Tortoise and Achilles. (Achilles was the great Greek hero of Homer's *The Illiad*.) It has inspired many writers and thinkers through the ages, notably Lewis Carroll and Douglas Hofstadter, who also wrote dialogues involving the Tortoise and Achilles.

The original goes something like this:



The Tortoise challenged Achilles to a race, claiming that he would win as long as Achilles gave him a small head start. Achilles laughed at this, for of course he was a mighty warrior and swift of foot, whereas the Tortoise was heavy and slow.

“How big a head start do you need?” he asked the Tortoise with a smile.

“Ten meters,” the latter replied.

Achilles laughed louder than ever. “You will surely lose, my friend, in that case,” he told the Tortoise, “but let us race, if you wish it.”

“On the contrary,” said the Tortoise, “I will win, and I can prove it to you by a simple argument.”

“Go on then,” Achilles replied, with less confidence than he felt before. He knew he was the superior athlete, but he also knew the Tortoise had the sharper wits, and he had lost many a bewildering argument with him before this.

“Suppose,” began the Tortoise, “that you give me a 10-meter head start. Would you say that you could cover that 10 meters between us very quickly?”

“Very quickly,” Achilles affirmed.

“And in that time, how far should I have gone, do you think?”

“Perhaps a meter – no more,” said Achilles after a moment's thought.

“Very well,” replied the Tortoise, “so now

there is a meter between us. And you would catch up that distance very quickly?"

"Very quickly indeed!"

"And yet, in that time I shall have gone a little way farther, so that now you must catch that distance up, yes?"

"Ye-es," said Achilles slowly.

"And while you are doing so, I shall have gone a little way farther, so that you must then catch up the new distance," the Tortoise continued smoothly.

Achilles said nothing.

"And so you see, in each moment you must be catching up the distance between us, and yet I – at the same time – will be adding a new distance, however small, for you to catch up again."

"Indeed, it must be so," said Achilles wearily.

"And so you can never catch up," the Tortoise concluded sympathetically.

"You are right, as always," said Achilles sadly – and conceded the race.

Zeno's Paradox may be rephrased as follows. Suppose I wish to cross the room. First, of course, I must cover half the distance. Then, I must cover half the remaining distance. Then, I must cover half the remaining distance. Then I must cover half the remaining distance . . . and so on forever. The consequence is that I can never get to the other side of the room.

What this actually does is to make *all* motion impossible, for before I can cover half the distance I must cover *half* of half the distance, and before I can do that I must cover half of half of half of the distance, and so on, so that in reality I can never move any distance at all, because doing so involves moving an infinite number of small intermediate distances first.

Now, since motion obviously *is* possible, the question arises, what is wrong with Zeno? What is the "flaw in the logic?" If you are giving the matter your full attention, it should begin to make you squirm a bit, for on its face the logic of the situation seems unassailable. You shouldn't be able to cross the room, and the Tortoise should win the race! Yet we know better. Hmm.

Rather than tackle Zeno head-on, let us pause to notice something remarkable. Suppose we take Zeno's Paradox at face value for the moment, and agree with him that before I can walk a mile I must first walk a half-mile. And before I can walk the remaining half-mile I must first cover half of it, that is, a quarter-mile, and then an eighth-mile, and then a sixteenth-mile, and then a thirtysecondth-mile, and so on. Well, suppose I could cover all these infinite number of small distances, how far should I have walked? One mile! In other words,

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

At first this may seem impossible: adding up an infinite number of positive distances should give an infinite distance for the sum. But it doesn't – in this case it gives a finite sum; indeed, all these distances add up to 1! A little reflection will reveal that this isn't so strange after all: if I can divide up a finite distance into an infinite number of small distances, then adding all those distances together should just give me back the finite distance I started with. (An infinite sum such as the one above is known in mathematics as an infinite series, and when such a sum adds up to a finite number we say that the series is *summable*.)

Now the resolution to Zeno's Paradox is easy. Obviously, it will take me some fixed time to cross half the distance to the other side of the room, say 2 seconds. How long will it take to cross half the remaining distance? Half as long – only 1 second. Covering half of the remaining distance (an eighth of the total) will take only half a second. And so one. And once I have covered all the infinitely many sub-distances and added up all the time it took to traverse them? Only 4 seconds, and here I am, on the other side of the room after all.

And poor old Achilles would have won his race.

ADDENDUM

So that you don't get to feeling too complacent about infinities in the small, here's a similar paradox for you to take away with you.

THOMPSON'S LAMP: Consider a lamp, with a switch. Hit the switch once, it turns it on. Hit it again, it turns it off. Let us imagine there is a being with supernatural powers who likes to play with this lamp as follows. First, he turns it on. At the end of one minute, he turns it off. At the end of half a minute, he turns it on again. At the end of a quarter of a minute, he turns it off. In one eighth of a minute, he turns it on again. And so on, hitting the switch each time after waiting exactly one-half the time he waited before hitting it the last time. Applying the above discussion, it is easy to see that all these infinitely

many time intervals add up to exactly two minutes.

QUESTION: At the end of two minutes, is the lamp on, or off?

ANOTHER QUESTION: Here the lamp started out being off. Would it have made any difference if it had started out being on?