Al-Khwarizmi: The Father of Algebra

Historical Background

Abu Ja'far Muhammad ibn Musa al-Khwarizmi was born circa 780 CE. Although his name indicates that his family was originally from the region Khwarizm near the Aral Sea, historians believe that al-Khwarizmi was born in the city of Baghdad in present day Iraq (Calinger, 199). While little is known about his private life, al-Khwarizmi's work and contributions to mathematics have largely survived the ages relatively intact. The exception is a book of arithmetic in which the original cannot be found; there is, however, a Latin translation of this work as well as other Arab references that cite the missing treatise. Al-Khwarizmi was a member of the House of Wisdom in Baghdad, a society established by the caliph for the study of science (Al-Daffa, 23). According to Al-Daffa, during al-Khwarizmi's life, much of the area between the Mediterranean and India was ruled by al-Mamun, an Islamic caliph who had consolidated his position in a protracted civil war. After pacifying the area under his control, al-Mamun became a patron of the sciences. He instituted the House of Wisdom to both translate the works of Byzantine and Greek scientists as well as to conduct research into various realms of science. Al-Mamun also built a library in Baghdad to house these works; this was the first large collection of scientific information constructed since the Library of Alexandria's erection several centuries before. Finally, al-Mamun constructed a lavish astronomical observatory in Baghdad for the use of Muslim astronomers. Within a short period of time, Baghdad became the new center for learning in the Mediterranean world (Al-Daffa, 23-34). This interest in Greek Hellenistic thought represented a tremendous change from previous Islamic ideology. This might lead one to ask why such seemingly sensible steps represent such a rapid departure from Islamic thought as well as what was the impetus for such a dramatic change?

The first idea to consider is that there had always been a fundamental difference from Greek and Islamic thought. The most important difference was a matter of religion. The classical Greeks and Romans believed in many Gods and the later, after Rome had Christianized the Mediterranean, they believed in a Holy Trinity (Smith, 340). These ideas directly conflicted with the Islamic belief of the one true God, Allah (Smith, 222). As a result, in the seventh century CE, when the disciples of Mohammed began their conquest of the Middle East, North Africa and Spain, the Muslims destroyed much of the work and knowledge of those that they conquered (Smith, 230). Their extreme Islamic fundamentalism blinded the

Arabs to the advanced scientific contributions of their neighbors. The initial conquests of Islam lasted well into the eighth century CE, just a generation or two prior to the birth of al-Khwarizmi and al-Mamun. Therefore, as a matter of time, al-Khwarizmi and al-Mamun are not far removed from the zealous invaders of the past.

The drastic change in Islamic attitudes toward western science might be a byproduct of the religion itself. Muslims live their lives according to the rules and precepts set forth in the *Qu'ran* (Koran). This book dictates all aspects of a Muslim's life and death. For example, the *Qu'ran* dictates that Muslims must pray several times a day toward the city of Mecca as well as giving precise rules of inheritance when one dies (Smith, 236). Both of these tasks require advanced knowledge of mathematics. Mathematics is used in the study of cartography, astronomy and geography. Knowledge of astronomy would have been critical for determining which direction to pray or for ascertaining the beginning of Ramadan (which is based largely on the phases of the moon). Other, less concrete, applications of math would have been required in order to properly divide up estates (Berggren, 63). In a sense, after the zeal of Islam aided in the destruction of knowledge, it realized just how useful that knowledge might have been for its own purposes. As a result, al-Mamun created the House of Wisdom to restore and research the answers to the scientific questions that plagued the administration of his empire. Al-Khwarizmi entered the House of Wisdom and, along with his colleagues, the Banu Musa, proceeded to travel and conduct research in geometry, astronomy and other areas of science. Along the way, al-Khwarizmi would publish several texts, however two texts on mathematics would lay the foundation for both the Arabic number system and the science of algebra.

A Journey to India and the work of Brahmagupta

One of the most important contributions of al-Khwarizmi's life was his role in the creation of the Arabic number system. In Europe, the popular number system was the clumsy Roman numeral system. Based on letters of the alphabet, Roman numerals made simple calculations difficult and complex calculations all but impossible. Most European and Middle Eastern number systems of the time were awkward and clumsy (*Arab Civilization-Sciences*). However, while traveling in India, al-Khwarizmi would come across a number system that would become the template for the modern Arabic number system. This system was largely based on the work of Brahmagupta (O'Connor and Robertson).

According to O'Connor and Robertson, Brahmagupta (598-670 CE) was a mathematician from Ujjain, India. Like al-Khwarizmi, Brahmagupta was an astronomer and mathematician for the court of the ruling family. Perhaps his most important contributions to mathematics were the creation of a decimal-based number system and the concept of zero. The zero, called sunya, was defined as subtracting any number from itself. In his work, *Brahmasphutasiddhanta*, Brahmagupta gives some properties of the new number, which included, "when zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero." Brahmagupta also tried to introduce the concepts of positive and negative numbers as fortunes and debts (O'Connor and Robertson). However, although al-Khwarizmi was able to accept zero, the idea of negative roots was still not prevalent in Al-Khwarizmi's work.

Al-Khwarizmi would extend upon the ideas of Brahmagupta. The Arabic number system would be similar to the Indian number system with ten digits 0-9 as well as similar treatment of mathematical operations like multiplication. However, al-Khwarizmi went further becoming one of the first mathematicians to use zero as a placeholder in positional base notation (*Mathematics and Astronomy)*. Although revolutionary, this idea was met with great skepticism. First, many mathematicians even as late as Renaissance Europe believed zero to be a "worthless nothing." Second, there were also those that could accept zero, but could not accept that attaching it to another number would somehow increase the numbers value tenfold. Even though it would take centuries for the world to accept zero, al-Khwarizmi had produced a number system similar to the one used worldwide today (*Mathematics and Astronomy*). The main differences were al-Khwarizmi's skepticism of the existence negative numbers and the difference between al-Khwarizmi's symbols and the modern Arabic numbers (it would take several centuries of evolution before numerals began to take a form familiar to the twenty-first century reader). However, it is difficult to compare more details about al-Khwarizmi's beliefs regarding number theory and arithmetic to today's number system. Unfortunately much of his work on the subject has been either lost or destroyed.

Much of what is known about al-Khwarizmi's work comes from secondary sources. For instance, another Arab mathematician Abu Mansur ibn Tahir al-Baghdadi cites al-Khwarizmi's lost work in his own work on number theory (O'Connor and Robertson). According to O'Connor and Robertson, al-Baghdadi (c.980-1037 CE) was an algorist, or a mathematician who principally continued the work of al-Khwarizmi.

Another source that relies on al-Khwarizmi's lost writings is the twelfth century Latin work *Algoritmi de numero Indorum* (in English *Al-Khwarizmi on the Hindu Art of Reckoning*). This text gives assigns the credit for the creation of the modern number system to al-Khwarizmi and his work based on Hindu mathematics (O'Connor and Robertson). Although never actually recovered, al-Khwarizmi's work on the creation of the Arabic number system is considered one of his two most important works. It is also due to the Latin translation of his name, Algoritmi, that the world received the word algorithm.

The Development of Algebra

If introducing the world to the Arabic number system were the only accomplishment that al-Khwarizmi would have produced in his life, this would still be sufficient to rank him as one of the world's great mathematicians. However, the prolific mathematician had an equally important contribution that he would put forth. Entitled *Kitbag al-mukhtasar fi hisab aj-jabr wa'l muqabalah* (in English, *The Condensed Book on Calculation by Restoring and Balancing*) (Berggren, 7-8). According to Berggren, the world al-jabr meaning to "restore" or "complete" would evolve into what the world knows as algebra. The other mathematical operation that al-Khwarizmi dealt with was muqabalah, which translates into the English word "reduction" or balancing (Berggren, 6-7). According to O'Connor and Robertson, in his work he both attempted to classify and solve various kinds of quadratic equations as well as give geometric paradigms for the operations. It is commonly believed that al-Khwarizmi was influenced by Greek, neo-Babylonian and Indian sources with the Indians supplying the number system, the Babylonians supplying the numerical processes and the Greeks supplying the tradition of rigorous proof (O'Connor and Robertson).

According to al-Khwarizmi, there are three kinds of quantities: **simple numbers** like 4 or 77, then the **root**, which is the unknown, x, to be found in a particular problem, and the **mal** (in English, wealth) which is the square of the root in the problem (Al-Daffa, 55). With these definitions of quantities, al-Khwarizmi classified problems into six standard forms (a, b and c are all positive numbers):

1. Squares equal to roots. Example: $ax^2 = bx$
2. Squares equal to numbers. Example: $ax^2 = b$
3. Roots equal to numbers. Example: $ax = b$
4. Squares and roots equal to numbers. Example: $ax^2 + bx = c$
5. Squares and numbers equal to roots. Example: $ax^2 + c = bx$
6. Roots and numbers equal to squares. Example: $ax^2 = bx + c$,

While any student of modern mathematics would recognize that all of al-Khwarizmi's different forms are merely different ways of expressing the general quadratic ($ax^2 + bx + c = 0$), al-Khwarizmi was unable to accept the existence for negative numbers. This made it a necessity for al-Khwarizmi to partition what modern mathematicians consider a subset of quadratic equations into several smaller subsets that were more congruent with the Arabic view of mathematics. The chapters of al-Khwarizmi's work roughly correspond to the different classes of quadratic equations. Al-Khwarizmi also had a very narrow view of what constituted a solution to a quadratic equation. Modern mathematics dictates that the quadratic equation has two solutions. They can have two real roots, one real root or two complex roots. However, for al-Khwarizmi's purposes, he was only interested in answers that had positive real roots. This would severely restrict the number of equations that al-Khwarizmi would deal with.

However, even with his restricted view of quadratic equations, al-Khwarizmi was able to make an important contribution regarding the existence of multiple roots of a quadratic equation. In particular, al-Khwarizmi came across this result while studying his fifth type of quadratic equation. Using only positive numbers, the only equation that could possibly yield two roots is the squares and numbers equal to roots. Al-Khwarizmi noticed the existence of two roots and illustrated it with the example $x^2 + 21 = 10x$ (Conger, Schorer and Overbay). According to the quadratic equation, x is equal to three or seven. By experimenting with different values and different equations of the same type, al-Khwarizmi was able to demonstrate the existence of multiple roots for quadratic equations. Thus al-Khwarizmi was able to show that a quadratic equation can have more than one solution (Conger, Overbay and Schorer).

Al-Khwarizmi's Geometric Proofs and Paradigms

Much of al-Khwarizmi's work with algebra drew heavily on concepts of geometry. In many of his proofs and paradigms, Khwarizmi represents simple numbers and roots as lengths of line segments. The multiplication of roots and numbers represented particular rectangles where the roots and numbers corresponded to the side lengths of the rectangles and their products represented the area of the rectangle. Familiar mathematic terms like "completing the square" or "squaring a polynomial" come from al-Khwarizmi's work with geometrically expressing algebraic expressions (Hollingdale, 97-98). For example, al-Khwarizmi solves $x^2 + 10x = 39$, by completing the square. The following is the geometric representation of the solution:

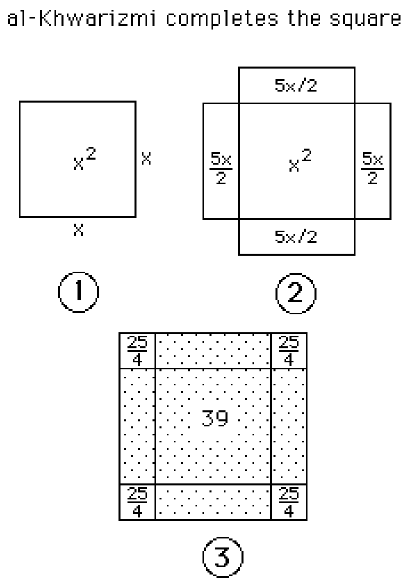

Fig 1—Completing the Square (O'Connor and Robertson)

In this example, al-Khwarizmi begins by constructing a square with side lengths x and area x^2 (picture 1). He then constructs four rectangles with side lengths x and 5/2 and area 5x/2 (picture 2). This polygon represents the $x^2 + 10x$ part of the equation. Al-Khwarizmi then completes the large square by adding four smaller squares with side lengths 5/2 and area 25/4 (picture 3). The total area of the four smaller squares is 25. On his equation, al-Khwarizmi adds 25 to both sides. Therefore the equation $x^2 + 10x = 39$ become $x^2 + 10x + 25 = 39 + 25$. Simplifying the expression yields $(x + 5)^2 = 64$, which implies that $x + 5 = 8$ and therefore $x = 3$ (O'Connor and Robertson). This example clearly indicates why al-Khwarizmi had difficulties accepting negative roots and coefficients. Because his numbers represented concrete quantities such as length or area, it would be impossible to create a negative area or length. Without an ability to construct those numbers, al-Khwarizmi refused to recognize their existence.

The previous example is indicative of the kind of solution that al-Khwarizmi found for his fourth class of equations, which he thoroughly discusses in the Chapter IV of *Al-jabr wa'l muqabalah*. His fifth and sixth classes of equations, which he discusses in Chapters V and VI respectively, required far more complex geometric representations. For example, take the equation $x^2 + 21 = 10x$, one of al-Khwarizmi's more famous examples. The following geometric model was used to find one of the solutions to this particular equation:

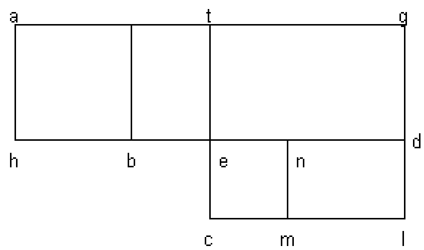

Fig 2—Class 5 Solution (Conger, Overbay and Schorer)

In this proof, the square *ab* represents x^2 and the rectangle *bg* represents 21 units. Then the large rectangle, comprising the square and the rectangle *bg* , must have an area equal to 10x, so that the side *ag* or *hd* must be 10 units. If then, one bisects *hd* at *e* , draws *et* perpendicular to *hd* , extends *te* to *c* so that *tc* = *tg* , and completes the squares *tclg* and *cmne* , the area *tb* is equal to area *md* . But *tl* is 25, and the gnomon *tenmlg* is 21 (since the gnomon is equal to the rectangle *bg*). Hence, the square *nc* is 4, and its side *ec* is 2. Inasmuch as *ec* = *be* , and since *he* = 5, we see that *x* = *hb* = 5 - 2 or 3 (Conger, Overbay and Schorer). Like the previous example, al-Khwarizmi constructed a geometric representation of the problem and then proceeded to complete a square, which held the key to his arriving at an answer. Although none of al-Khwarizmi's work contained the same rigor found in Greek mathematics, his geometric proofs and paradigms represented a concrete representation of algebra that would lay the foundation for the study of algebra in both the Arab world and in Europe.

The Practical Uses of al-Khwarizmi's Mathematics and Other Contributions

As explained earlier in the text, much of the House of Wisdom's work and research was directed toward a practical end. With the introduction of the new Hindu-based number system, al-Khwarizmi made calculations of all kinds substantially easier than previous number systems. However, because of his use of zero, particularly its role as a placeholder, the general public was highly skeptical of using his new number system. It took several centuries for Arabic number system to gain wide acceptance. Al-Khwarizmi, though, was able to use his new arithmetic and algebra to simplify the calculating of inheritance. Even today, many of the inheritance laws in Arab countries are based on the inheritance laws outline in the *Qu'ran*. This calls for an official to divide up the deceased person's possessions according to certain proportions based on the relationship of the beneficiary to the deceased (*Mathematics and Astronomy*). Using al-Khwarizmi's new methods of calculation and geometric representation, the local governments were better able to handle the affairs of the deceased. According to The Free Arab Voice:

> *Because of the Qur'an's very concrete prescriptions regarding the division of an estate among children of a deceased person, it was incumbent upon the Arabs to find the means for very precise delineation of lands. For example, let us say a father left an irregularly shaped piece of land-seventeen acres large-to his six sons, each OAA~ of whom had to receive precisely one-sixth of his legacy. The mathematics that the Arabs had inherited from the Greeks made such a division extremely complicated, if not impossible. It was the search for a more accurate, more comprehensive, and more flexible method that led Khawarazmi*(sic) *to the invention of algebra.* (*Mathematics and Astronomy*)

Al-Khwarizmi also made other contributions as a member of the House of Wisdom. In one of his works pertaining to astronomy, al-Khwarizmi constructed a sine table with base 150, a common Hindu parameter (Calinger, 200). Al-Khwarizmi was also part of a team that attempted to measure the circumference of the Earth.

Perhaps al-Khwarizmi's greatest contribution to mathematics was the debate his work inspired. Sometime after his death, the scientific community split between two groups, the algorists, who were proponents of the al-Khwarizmi's Hindu-based number system, and those who renounced ideas like zero as a number or placeholder. Al-Khwarizmi's work also laid the foundation for future Arab and European mathematicians. After al-Khwarizmi, other mathematicians built on his concrete ideas of algebra and his number theory. Some would introduce the numerals that are used all over the world today. Others would begin to find new classes of equations and their solutions. In the end, al-Khwarizmi's work would become the foundation of modern mathematics, thus earning al-Khwarizmi the title of "the Father of Algebra."

Epilogue

Much of al-Khwarizmi's work was inspired by his travels, especially eastward to India. It is perhaps fitting then that another traveler would travel east to learn and spread Arab algebra to Europe. According to Calinger, Leonardo of Pisa (c. 1170-1240 CE, also known as Leonardo Fibonacci) was the son of Guilielmo Bonacci, a Pisan bureaucrat. As the director of a Pisan trading colony in Bugia, Algeria, Bonacci understood the quality of instruction Leonardo would receive in Arab schooling. As a result, Leonardo was set to study in various Arabic cities. Leonardo would also run work for his father in various cities in the Arab world. Having studied various means of calculation, Fibonacci deemed the Hindu-Arabic number system to be the superior vehicle for calculations. Between 1200 and 1225 CE, Fibonacci produced his works *Liber abbaci* (in English, *Book of the Abacus*)*, Pracitca geometriae* (in English, *Practice of Geometry)* and *Liber quadratorum* (in English, *Book of Square Numbers*). His work drew heavily upon Arab concepts and number theory. These works, especially *Liber abbaci* brought Fibonacci to the attention of Holy Roman Emperor Frederick II, who invited Fibonacci to his court in Pisa (Calinger, 250-251). This represented the endorsement that was needed for Arab mathematics to take hold in Europe. As a result, the numbers, arithmetic and algebra that are used worldwide today can trace their origins to the work of Abu Ja'far Muhammad ibn Musa al-Khwarizmi and the House of Wisdom in Baghdad.

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